Barvinok on Parametrized Polyhedra

Sven Verdoolaege

0.0%

Overview

- Introduction
- Non-parametrized Barvinok
- Polytope Reduction
- Implementation
- Parametrized Barvinok
- Large Periods

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Motivation

```
POLYHEDRON Dimension:12
           Constraints:23  Equations:0  Rays:3072  Lines:0
Constraints 23 14
Inequality: [   0   0   0  -1   0   0   0   0   0   0   0   0   8 ]
Inequality: [   0   1   0   0   0   0   0   0   0   0   0   0   0 ]
Inequality: [   0   0   0   0  -1   0   0   0   0   0   0   0   3 ]
Inequality: [   0   0   0   0   0  -1   0   0   0   0   0   0   3 ]
Inequality: [   0   0   1   0   0   0   0   0   0   0   0   0   0 ]
Inequality: [   0   0  -1   0   0   0   0   0   0   0   0   0   8 ]
Inequality: [   0  -1   0   0   0   0   0   0   0   0   0   0   8 ]
Inequality: [   0   0   0   0   0   1   0   0   0   0   0   0   0 ]
Inequality: [   0   0   0   1   0   0   0   0   0   0   0   0   0 ]
Inequality: [   0   0   0   0   1   0   0   0   0   0   0   0   0 ]
Inequality: [   1   0   0   0   0   0   0   0   0   0   0   0   0 ]
Inequality: [   0   0   0   0   0   0   0   0   0  -1   0   0   8 ]
Inequality: [   0   0   0   0   0   0  -1   0   0   0   0   0  10 ]
Inequality: [   0   0   0   0   0   0   0   1   0   0   0   0   0 ]
Inequality: [   0   0   0   0   0   0   0   0   0   0  -1   0   3 ]
Inequality: [   0   0   0   0   0   0   0   0   0   0   0  -1   3 ]
Inequality: [   0   0   0   0   0   0   0   0   1   0   0   0   0 ]
Inequality: [   0   0   0   0   0   0   0   0  -1   0   0   0   8 ]
Inequality: [   0   0   0   0   0   0   0  -1   0   0   0   0   8 ]
Inequality: [   0   0   0   0   0   0   0   0   0   0   0   1   0 ]
Inequality: [   0   0   0   0   0   0   0   0   0   1   0   0   0 ]
Inequality: [   0   0   0   0   0   0   0   0   0   0   1   0   0 ]
Inequality: [  -1   0   0   0   0   0   1   0   0   0   0   0  -1 ]
Rays 3072 14
...
```

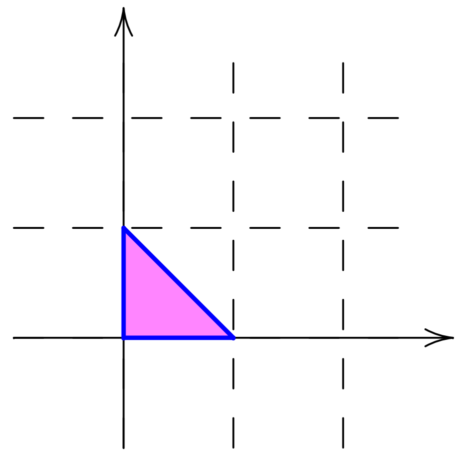
6.4%

Motivation

- Number of lattice points: 7482689280
- Time to compute (“manual”): over 4 hours
- Time to compute (Barvinok): 50s

9.6%

Running example

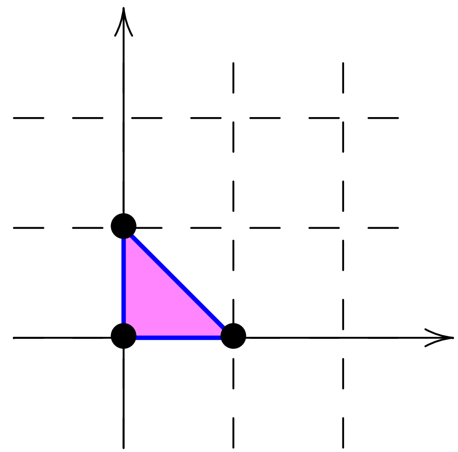


$$T = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} | \exists \lambda_i : \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

$$T = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} | x \geq 0 \wedge y \geq 0 \wedge x + y \leq 1 \right\}$$

11.2%

Running example



$$T = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} | \exists \lambda_i : \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

$$T = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} | x \geq 0 \wedge y \geq 0 \wedge x + y \leq 1 \right\}$$

Number of lattice points: 3

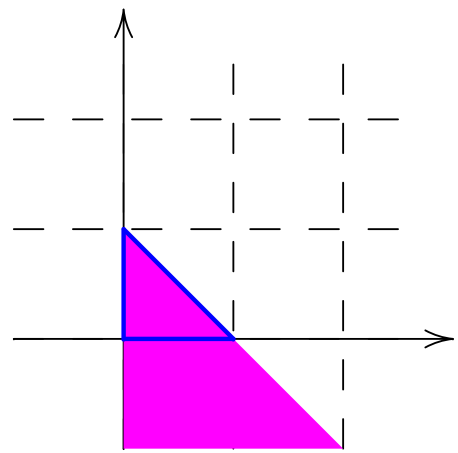
12.9%

Supporting cone

$$\text{cone}(P, F) = \left\{ \mathbf{x} \in \mathbb{R}^d | \mathbf{a}_i^T\mathbf{x} \geq \mathbf{b}_i \text{ for } i \in I_F \right\}$$

where I_F is the set of inequalities active on face F.

E.g, supporting cone of T at $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$:



16.1%

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Barvinok's Algorithm

1. For each vertex $\mathbf{v}_i$ of P
   (a) Determine supporting cone $\mathrm{cone}(P, \mathbf{v}_i)$
   (b) Let $K = \mathbf{x}^{-\mathbf{v}_i} \, \mathrm{cone}(P, \mathbf{v}_i)$
   (c) Decompose K into unimodular cones K_j
   (d) For each K_j
      i. Determine $f(K_j; \mathbf{x})$
   (e) $f(K; \mathbf{x}) = \sum_j \epsilon_j f(K_j; \mathbf{x})$
   (f) $f(\mathrm{cone}(P, \mathbf{v}_i); \mathbf{x}) = \sum_j \epsilon_j \mathbf{x}^{\mathcal{E}'(\mathbf{v}_i, K_j)} f(K_j; \mathbf{x})$
2. $f(P; \mathbf{x}) = \sum_i f(\mathrm{cone}(P, \mathbf{v}_i); \mathbf{x})$
3. evaluate $f(P; \mathbf{1})$

19.3%

Evaluate Generating Function

$$f(P;\mathbf{x}) = \sum_i \sum_k \epsilon_{ik} \frac{\mathbf{x}^{\mathbf{v}_i}}{\prod_j \left(1 - \mathbf{x}^{\mathbf{u}_{ikj}}\right)}$$

We want to evaluate $f(P;\mathbf{1})$

- Multivariate $\to$ univariate $x_i = t^{\mu_i}$
- Remove negative powers
- Substitute t by $s+1$
- Compute residue

22.5%

Multivariate $\rightarrow$ univariate

- Find $\boldsymbol{\mu}$ such that

$$\forall i, k, j : \boldsymbol{\mu}^T \mathbf{v}_i \neq 0 \wedge \boldsymbol{\mu}^T \mathbf{u}_{ikj} \neq 0$$

(using small random values for μ or on the moment curve)

- Substitute x_i by t^{μ}

$$\epsilon_{ik} \frac{\mathbf{x}^{\mathbf{v}_i}}{\prod_j \left(1 - \mathbf{x}^{\mathbf{u}_{ikj}}\right)} = \epsilon_{ik} \frac{t^{c_0}}{\prod_{j=1}^{d} \left(1 - t^{c_j}\right)}$$

with $c_0 = \boldsymbol{\mu}^T \mathbf{v}_i$ and $c_j = \boldsymbol{\mu}^T \mathbf{u}_{ikj}$.

- Evaluate f at $t = 1$

25.8%

Multivariate → univariate

$$\frac{y}{(1-y^{-1})(1-xy^{-1})} + \frac{x}{(1-x^{-1})(1-x^{-1}y)} + \frac{1}{(1-x)(1-y)}$$

Take $\boldsymbol{\mu} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, i.e., $x = t$ and $y = t^{-1}$

$$\frac{t^{-1}}{(1-t)(1-t^2)} + \frac{t}{(1-t^{-1})(1-t^{-2})} + \frac{1}{(1-t)(1-t^{-1})}$$

29.0%

Remove negative powers

- Let
$$p := |\{c_j | c_j > 0\}|$$
- Let
$$c := -\left(\sum_{1\le j\le d, c_j<0} c_j\right)$$
-
$$\epsilon_{ik}\frac{t^{c_0}}{\prod_{j=1}^{d}(1-t^{c_j})} = \epsilon'_{ik}\frac{t^{c_0+c}}{\prod_{j=1}^{d}(t^{c_j}-1)} = \epsilon'_{ik}t^{c^*}\frac{t^{c_0+c-c^*}}{\prod_{j=1}^{d}(t^{c_j}-1)}$$
with $\epsilon'_{ik} = (-1)^p\epsilon_{ik}$ and $c^* = \min_{ik}(c_0+c)$

32.2%

Remove negative powers

$$\frac{t^{-1}}{(1-t)(1-t^2)} + \frac{t}{(1-t^{-1})(1-t^{-2})} + \frac{1}{(1-t)(1-t^{-1})}$$

$$\Downarrow$$

$$\frac{1}{t}\left(\frac{1}{(t-1)(t^2-1)} + \frac{t^5}{(t-1)(t^2-1)} - \frac{t^2}{(t-1)(t-1)}\right)$$

35.4%

Substitute t by $s+1$

- $$\epsilon'_{ik} t^{c^*} \frac{t^{c_0+c-c^*}}{\prod_{j=1}^{d} \left(t^{c_j}-1\right)} = \epsilon'_{ik} (s+1)^{c^*} \frac{P(s)}{\prod_{j=1}^{d} s Q_j(s)}$$
- Evaluate for $s=0$
- $\Rightarrow$ find coefficient of s^d in

$$\frac{P(s)}{\prod_{j=1}^{d} Q_j(s)}$$

38.7%

Substitute t by $s+1$

$$\frac{1}{t}\left(\frac{1}{(t-1)(t^2-1)}+\frac{t^5}{(t-1)(t^2-1)}-\frac{t^2}{(t-1)(t-1)}\right)$$

$$\Downarrow$$

$$\frac{1}{s+1}\left(\frac{1}{s^2(s+2)}+\frac{(s+1)^5}{s^2(s+2)}-\frac{(s+1)^2}{s^2}\right)$$

$$\frac{1}{s+1}\frac{1}{s^2}\left(\left(\frac{1}{8}s^2-\frac{1}{4}s+\frac{1}{2}\right)+\left(\frac{31}{8}s^2+\frac{9}{4}s+\frac{1}{2}\right)-\left(s^2+2s+1\right)\right)$$

(modulo s^3)

41.9%

Compute Residue

$$\frac{1}{s+1}\frac{1}{s^2}\left(\left(\frac{1}{8}s^2-\frac{1}{4}s+\frac{1}{2}\right)+\left(\frac{31}{8}s^2+\frac{9}{4}s+\frac{1}{2}\right)-(s^2+2s+1)\right)$$

$$1.\left(\frac{1}{8}+\frac{31}{8}-1\right)=3$$

45.1%

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45.1%

Polytope reduction

Assume

$$P = P_1 \oplus P_2$$

then

$$\#P = \#P_1 \cdot \#P_2$$

Special case:

$$P = [l, u] \oplus P_2$$

$$\#P = (\lfloor u \rfloor - \lceil l \rceil + 1) \cdot \#P_2$$

Motivational example reduces to

$$136048896 \cdot \#\{(x, y) | -x + y - 1 \geq 0, -y + 10 \geq 0, x \geq 0\}$$

Computation time: 5s

48.3%

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Implementation

1. Handle special cases
2. Remove equalities
3. Reduce polytope
4. for each vertex
   (a) compute supporting cone
   (b) decompose supporting cone
5. compute non-orthogonal vector μ
6. for each vertex
   (a) for each cone in decomposition
      i. compute numerator
      ii. “normalize” rational function
7. factor out common factor in numerator
8. for each vertex
   (a) for each cone in decomposition
      i. `count +=` residue

51.6%

Implementation

- Remove equalities

$$\mathbf{a}^T\mathbf{x} = c$$

  - $\gcd(\mathbf{a}) \nmid c$
    $\Rightarrow$ count = 0
  - $\gcd(\mathbf{a}) \mid c$
    $\Rightarrow$ transform P by unimodular U with first row $\mathbf{a}^T$
    $\Rightarrow$ drop first dimension

(special case of reduction)

54.8%

Implementation

- Special cases
  - Empty $\Rightarrow$ 0
  - Unbounded $\Rightarrow$ -1
- compute supporting cone
  $\Rightarrow$ straightforward implementation
- decompose supporting cone
  $\Rightarrow$ regular triangulation + Barvinok
- compute non-orthogonal vector μ
  $\Rightarrow$ small random values
  $\Rightarrow$ fall-back case based on the moment curve not implemented yet

58.0%

Implementation

- compute numerator
  $\Rightarrow \mathcal{E}(\mathbf{v}, R, \boldsymbol{\mu})$ = `lattice_point(v, R, mu)`

$$R = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_d \end{bmatrix} \in \mathbb{Z}^{d,d}$$

$$\boldsymbol{\lambda} = \frac{1}{m} R^{-T} \mathbf{w}$$

(with $\mathbf{w} = m\mathbf{v}$)

$$\mathcal{E}(\mathbf{v}, R, \boldsymbol{\mu}) = \boldsymbol{\mu}^T R^T \left\lceil \frac{1}{m} R^{-T} \mathbf{w} \right\rceil$$

61.2%

Implementation

- compute residue
  $\Rightarrow$ find coefficient of s^d in

$$\frac{P(s)}{\prod_{j=1}^{d} Q_j(s)} = \frac{(s+1)^{c_0+c-c^*}}{\prod_{j=1}^{d} \frac{(s+1)^{c_j}-1}{s}}$$

$$P'(s) = \sum_{i=0}^{\min(k,d)} \binom{k}{i} s^i \qquad Q'(s) = \sum_{i=1}^{\min(k,d+1)} \binom{k}{i} s^{i-1}$$

$$\binom{k}{i} = \frac{k-i+1}{i} \binom{k}{i-1}$$

64.5%

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64.5%

Parametrized Algorithm

1. For each vertex $\mathbf{v}_i(\mathbf{N})$ of P
   (a) Determine supporting cone $\mathrm{cone}(P, \mathbf{v}_i(\mathbf{N}))$
   (b) Let $K = \mathbf{x}^{-\mathbf{v}_i(\mathbf{N})} \, \mathrm{cone}(P, \mathbf{v}_i(\mathbf{N}))$
   (c) Decompose K into unimodular cones K_j
   (d) For each K_j
      i. Determine $f(K_j; \mathbf{x})$
   (e) $f(K; \mathbf{x}) = \sum_j \epsilon_j f(K_j; \mathbf{x})$
   (f) $f(\mathrm{cone}(P, \mathbf{v}_i(\mathbf{N})); \mathbf{x}) = \sum_j \epsilon_j \mathbf{x}^{\mathcal{E}'(\mathbf{v}_i(\mathbf{N}), K_j)} f(K_j; \mathbf{x})$
2. For each validity domain D of P
   (a) $f(P; \mathbf{x}) = \sum_{\mathbf{v}_i \in D} f(\mathrm{cone}(P, \mathbf{v}_i(\mathbf{N})); \mathbf{x})$
   (b) evaluate $f(P; \mathbf{1})$

67.7%

Implementation

1. Handle special cases
2. Remove equalities
3. for each vertex
   (a) compute supporting cone
   (b) decompose supporting cone
4. for each validity domain
   (a) compute non-orthogonal vector μ
   (b) for each vertex
      i. for each cone in decomposition
         A. compute numerator
         B. "normalize" rational function
   (c) factor out common factor in numerator
   (d) for each vertex
      i. for each cone in decomposition
         A. `count +=` residue

70.9%

Implementation

- Remove equalities

$$\mathbf{a}^T\mathbf{x} = \mathbf{b}^T\mathbf{N} + c$$

  - $\mathbf{a} = 0$
    $\Rightarrow$ leave for `Polyhedron2Param_SimplifiedDomain` to handle
  - $\mathbf{a} \neq 0$
    $\Rightarrow$ transform P by unimodular U with first row $\mathbf{a}^T$
    $\Rightarrow$ drop first dimension
    - $\gcd(\mathbf{a}) \nmid \gcd(\mathbf{b}, c)$
      $\Rightarrow$ multiply $E(\mathbf{N})$ by $(\mathbf{b}^T\mathbf{N} + c \bmod \frac{\gcd(\mathbf{a})}{\gcd(\mathbf{a},\mathbf{b},c)} = 0)$

74.1%

Implementation

- Special cases
  - Empty $\Rightarrow$ 0
  - Non-parametrized $\Rightarrow$ `barvinok_count`

- compute supporting cone
  $\Rightarrow$ straightforward implementation
  $\Rightarrow$ available in `Find_m_faces` ?

77.4%

Implementation

- compute numerator
  Vertices: $V \in \mathbb{Q}^{d,n+1}$; $W = mV \in \mathbb{Z}^{d,n+1}$

$$\Lambda = \frac{1}{m} R^{-T} W = \frac{1}{m} \bar{\Lambda}$$

$$\lceil \Lambda \rceil = \Lambda + \frac{(-\bar{\Lambda}) \bmod m}{m}$$

$$\begin{aligned} \tilde{V} &= R^T \Lambda + R^T \frac{(-\bar{\Lambda}) \bmod m}{m} \\ &= V + \frac{1}{m} R^T \left((-\bar{\Lambda}) \bmod m \right) \end{aligned}$$

80.6%

Implementation

- compute numerator

$$\tilde{V} = \frac{1}{m} R^T \left(\left(-\bar{\Lambda} \right) \bmod m \right)$$

$$\boldsymbol{\mu}^T \tilde{V} = \boldsymbol{\mu}^T V + \frac{1}{m} \boldsymbol{\mu}^T R^T \left(\left(-\bar{\Lambda} \right) \bmod m \right)$$

$$\begin{aligned} \tilde{\mathbf{v}}^T \mathbf{N} &= \mathbf{v}^T \mathbf{N} + \bar{\mathbf{v}}^T \mathbf{N} \\ &= \mathbf{v}^T \mathbf{N} + U_{\mathbf{N}} \end{aligned}$$

$$\mathcal{E}(V \mathbf{N}_i, R, \boldsymbol{\mu}) = \tilde{\mathbf{v}}^T \mathbf{N}_i$$

$$U_{\mathbf{N}_i} = \mathcal{E}(V \mathbf{N}_i, R, \boldsymbol{\mu}) - \mathbf{v}^T \mathbf{N}_i .$$

83.8%

Implementation

- compute residue

$$P(s) = (s+1)^{\mathbf{v}^T\mathbf{N}+U_\mathbf{N}} = (s+1)^{n+c}$$

$\Rightarrow$ coefficient of s^d in

$$\frac{P(s)}{\prod_{j=1}^{d} Q_j(s)}$$

is d-polynomial in $n = \mathbf{v}^T\mathbf{N} + U_\mathbf{N} - c$
$\Rightarrow$ evaluate using Horner

87.0%

Integer vertices

$\mathbf{v}(\mathbf{N})$ integer $\Rightarrow \mathbf{v}(\mathbf{N}) = \mathcal{E}'(\mathbf{v}_i(\mathbf{N}), K_j) \quad \forall K_j$

$$f(\text{cone}(P, \mathbf{v}_i(\mathbf{N})); \mathbf{x}) = \mathbf{x}^{\mathbf{v}_i(\mathbf{N})} \left(\sum_j \epsilon_j f(K_j; \mathbf{x}) \right)$$

$\Rightarrow$ single “parametrized multiplication”

Integer vertices

$\mathbf{v}(\mathbf{N})$ integer $\Rightarrow \mathbf{v}(\mathbf{N}) = \mathcal{E}'(\mathbf{v}_i(\mathbf{N}), K_j) \quad \forall K_j$

$$f(\text{cone}(P, \mathbf{v}_i(\mathbf{N})); \mathbf{x}) = \mathbf{x}^{\mathbf{v}_i(\mathbf{N})} \left(\sum_j \epsilon_j f(K_j; \mathbf{x}) \right)$$

$\Rightarrow$ single “parametrized multiplication”

Integer Hull ?

89.2%

Integer vertices

$\mathbf{v}(\mathbf{N})$ integer $\Rightarrow \mathbf{v}(\mathbf{N}) = \mathcal{E}'(\mathbf{v}_i(\mathbf{N}), K_j) \quad \forall K_j$

$$f(\mathrm{cone}(P, \mathbf{v}_i(\mathbf{N})); \mathbf{x}) = \mathbf{x}^{\mathbf{v}_i(\mathbf{N})} \left(\sum_j \epsilon_j f(K_j; \mathbf{x}) \right)$$

$\Rightarrow$ single “parametrized multiplication”

Integer Hull ?
$\Rightarrow$ “periodic” supporting cone
$\Rightarrow$ different decomposition for each value in the period

90.3%

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90.3%

Large periods

Large coefficients for parameters $\Rightarrow$ (possibly) large periods
Example:

```
9       7
#
#       dm      alpha   i       j       k       cst
1       -4      200     1       0       0       0
1       4       -200    -1      0       0       3
1       0       0       1       0       0       0
1       0       0       -1      0       0       199
1       0       0       0       1       0       -1
1       0       0       0       -1      0       199
1       0       0       0       0       1       0
1       -1      0       0       0       0       9999
1       4       0       -1      0       -200    -197
# nr_rows nr_columns Parameters
0       5
```

93.5%

Large periods—Ehrhart polynomial

```
( -1/200 * P + ( -1 * R + [ 199, 39801/200, 19901/100, 39803/200,
9951/50, 7961/40, 19903/100, 39807/200, 4976/25, 39809/200, 3981/20,
39811/200, 9953/50, 39813/200, 19907/100, 7963/40, 4977/25, 39817/200,
19909/100, 39819/200, 1991/10, 39821/200, 19911/100, 39823/200,
4978/25, 1593/8, 19913/100, 39827/200, 9957/50, 39829/200, 3983/20,
39831/200, 4979/25, 39833/200, 19917/100, 7967/40, 9959/50, 39837/200,
19919/100, 39839/200, 996/5, 39841/200, 19921/100, 39843/200, 9961/50,
7969/40, 19923/100, 39847/200, 4981/25, 39849/200, 797/4, 39851/200,
9963/50, 39853/200, 19927/100, 7971/40, 4982/25, 39857/200, 19929/100,
39859/200, 1993/10, 39861/200, 19931/100, 39863/200, 4983/25, 7973/40,
19933/100, 39867/200, 9967/50, 39869/200, 3987/20, 39871/200, 4984/25,
39873/200, 19937/100, 1595/8, 9969/50, 39877/200, 19939/100, 39879/200,
997/5, 39881/200, 19941/100, 39883/200, 9971/50, 7977/40, 19943/100,
39887/200, 4986/25, 39889/200, 3989/20, 39891/200, 9973/50, 39893/200,
19947/100, 7979/40, 4987/25, 39897/200, 19949/100, 39899/200, 399/2,
39901/200, 19951/100, 39903/200, 4988/25, 7981/40, 19953/100, 39907/200,
9977/50, 39909/200, 3991/20, 39911/200, 4989/25, 39913/200, 19957/100,
7983/40, 9979/50, 39917/200, 19959/100, 39919/200, 998/5, 39921/200,
19961/100, 39923/200, 9981/50, 1597/8, 19963/100, 39927/200, 4991/25,
39929/200, 3993/20, 39931/200, 9983/50, 39933/200, 19967/100, 7987/40,
4992/25, 39937/200, 19969/100, 39939/200, 1997/10, 39941/200, 19971/100,
39943/200, 4993/25, 7989/40, 19973/100, 39947/200, 9987/50, 39949/200,
799/4, 39951/200, 4994/25, 39953/200, 19977/100, 7991/40, 9989/50,
39957/200, 19979/100, 39959/200, 999/5, 39961/200, 19981/100, 39963/200,
9991/50, 7993/40, 19983/100, 39967/200, 4996/25, 39969/200, 3997/20,
39971/200, 9993/50, 39973/200, 19987/100, 1599/8, 4997/25, 39977/200,
19989/100, 39979/200, 1999/10, 39981/200, 19991/100, 39983/200, 4998/25,
7997/40, 19993/100, 39987/200, 9997/50, 39989/200, 3999/20, 39991/200,
4999/25, 39993/200, 19997/100, 7999/40, 9999/50, 39997/200, 19999/100,
39999/200 ]_P )
 )
```

96.7%

Large periods—Ehrhart polynomial

Or...

$$-\frac{P}{200} - R + 199 + \frac{P \bmod 200}{200}$$

98.3%

Large periods—Ehrhart polynomial

Or...

$$-\frac{P}{200} - R + 199 + \frac{P \bmod 200}{200}$$

$\Rightarrow$ Perform calculations with modulo expressions instead of periodic numbers

100.0%