Operations on Partitioned Ehrhart Polynomials and Reuse Distances for Caches with Limited Associativity

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Ehrhart Polynomials

$$\begin{aligned} S &= \mathbb{Z}^d \cap \{\mathbf{x} \in \mathbb{Q}^d | A\mathbf{x} + C\mathbf{p} + \mathbf{b} \geq 0\} \\ &= \mathbb{Z}^d \cap \left\{\mathbf{x} \in \mathbb{Q}^d | V(\mathbf{p})\boldsymbol{\nu}, \boldsymbol{\nu} \geq \mathbf{0}, \sum \nu = 1\right\} \end{aligned}$$

Vertex $V_j(\mathbf{p}) = \sum_i \lambda_{ji} p_i + f_j$

$\#S$ is an Ehrhart polynomial $\mathcal{E}(\mathbf{p})$

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Ehrhart Polynomials

$$\begin{aligned} S &= \mathbb{Z}^d \cap \{\mathbf{x} \in \mathbb{Q}^d | A\mathbf{x} + C\mathbf{p} + \mathbf{b} \geq 0\} \\ &= \mathbb{Z}^d \cap \left\{\mathbf{x} \in \mathbb{Q}^d | V(\mathbf{p})\boldsymbol{\nu}, \boldsymbol{\nu} \geq \mathbf{0}, \sum \nu = 1\right\} \end{aligned}$$

Vertex $V_j(\mathbf{p}) = \sum_i \lambda_{ji} p_i + f_j$

$\#S$ is an Ehrhart polynomial $\mathcal{E}(\mathbf{p})$

- Pseudo-polynomial of degree d in $\mathbf{p}$
- Coefficients: periodic numbers $u_{\mathbf{p}}$ with period $\mathbf{s}$

$$u_{\mathbf{p}} = \mathbf{u}[p_1 \bmod s_1][p_2 \bmod s_2] \ldots [p_n \bmod s_n]$$

- Period s_i is lcm of denominators of λ_{ji}
- Can be obtained through interpolation

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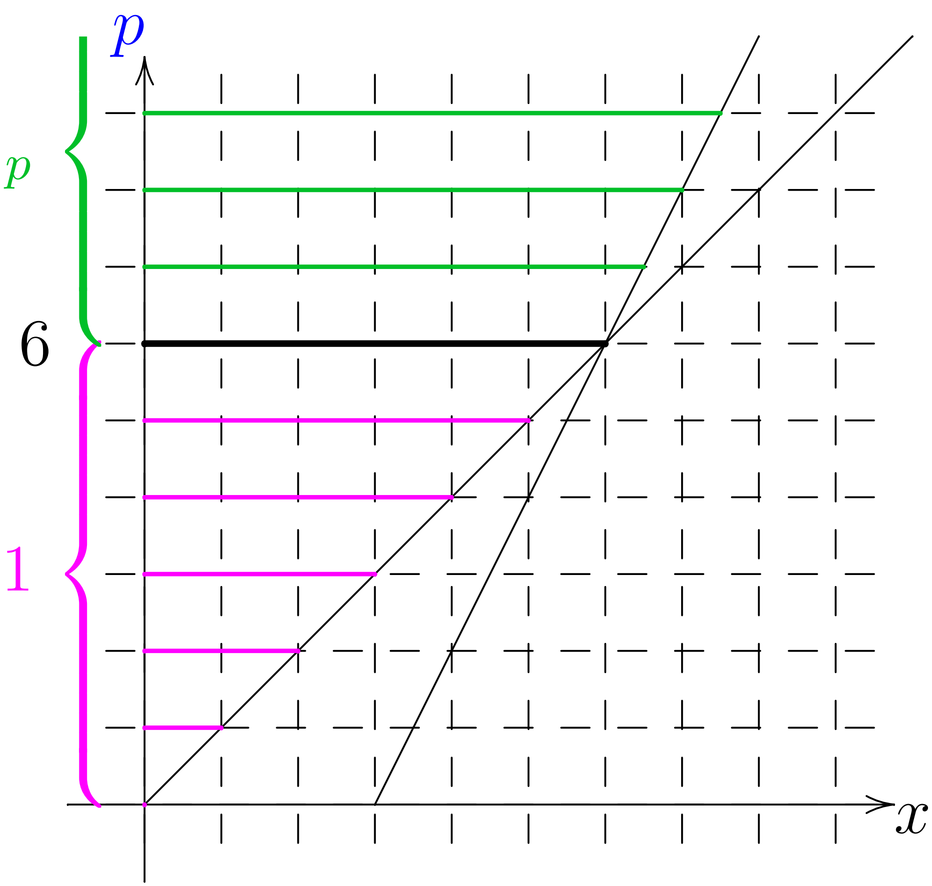
Validity domains

$$P = \{x | x \geq 0, 2x \leq p + 6, x \leq p\}$$

$$p \geq 6 \Rightarrow \mathcal{E}(p) = \frac{p}{2} + [4, \frac{7}{2}]_p$$

$$0 \leq p \leq 6 \Rightarrow \mathcal{E}(p) = p + 1$$



Forward Reuse Distance

- Reuse Pair: a pair of iterations that corresponds to subsequent accesses to the same memory element
- Accessed Data Set: the set of data accessed in between two iterations
- FRD: Number of memory elements read in between a memory reference at iteration $\mathbf{i}$ and the next reference to the same memory element, i.e., the number of elements in the accessed data set of the reuse pair with $\mathbf{i}$ as first element.

a	a	b	c	c	b	c	d	a	c
0	3	1	0	1	∞	2	∞	∞	∞

Reuse Pairs

$$\forall r, s \in \mathcal{R} : \text{reuse}_{r \to s} = \left\{ (\mathbf{i}_r, \mathbf{j}_s) \in \mathbb{Z}^{2n} : \text{subject to conditions (1a)–(1d)} \right\}$$

$$\mathbf{i}_r \in I_r \wedge \mathbf{j}_s \in I_s \qquad \textit{(iteration space)} \qquad (1a)$$

$$\mathbf{i}_r \prec \mathbf{i}_s \qquad \textit{(execution ordering)} \qquad (1b)$$

$$r@\mathbf{i}_r = s@\mathbf{j}_s \qquad \textit{(same location)} \qquad (1c)$$

$$\forall t \in \mathcal{R} : \neg \left(\exists \mathbf{k}_t \in I_t : \mathbf{i}_r \prec \mathbf{k}_t \prec \mathbf{j}_s \wedge t@\mathbf{k}_t = r@\mathbf{i}_r \right) \qquad \textit{(no intervening access)} \qquad (1d)$$

$$\text{reused}_{r \to s} = \{ \mathbf{i}_r \in I_r \mid \exists \mathbf{j}_s \in I_s : (\mathbf{i}_r, \mathbf{j}_s) \in \text{reuse}_{r \to s} \}$$

$\bigcup_{s \in \mathcal{R}} \text{reuse}_{r \to s}$ is a function; $\{\text{reused}_{r \to s}\}_s$ is a partition

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Forward Reuse Distance

$$\text{map}_r = \{\mathbf{i} \to l \in I_r \times L \mid l = r@\mathbf{i}\}$$

$$\text{iters}_{r\to t\to s} = \{\mathbf{i}_r, \mathbf{i}_s \to \mathbf{k}_t \in I_r \times I_s \times I_t : \mathbf{i}_r \prec \mathbf{k}_t \prec \mathbf{j}_s\}$$

$$\begin{aligned}\text{ADS}_{r\to s} = \{&\mathbf{i}_r, \mathbf{i}_s, l \in I_r \times I_s \times L \mid \\ &\exists t \in \mathcal{R}, \exists \mathbf{k}_t : \text{iters}_{r\to t\to s}(\mathbf{i}_r, \mathbf{i}_s, \mathbf{k}_t) \wedge \text{map}_t(\mathbf{k}_t, l)\}\end{aligned}$$

$$\text{FADS}_{r\to s} = \{\mathbf{i}_r, l \in \text{reused}_{r\to s} \times L \mid \exists \mathbf{j}_s \in I_s : (\mathbf{i}_r, \mathbf{j}_s, l) \in \text{ADS}_{r\to s}\}$$

$$\text{FRD}_{r\to s} : \text{reused}_{r\to s} \to \mathbb{Z} : \mathbf{i}_r \mapsto \left|\text{FADS}_{r\to s}\right| = \mathcal{E}\left(\text{FADS}_{r\to s}; \mathbf{i}_r\right)$$

$$\text{FRD}_r = \sum_{s \in \mathcal{R}} \text{FRD}_{r\to s}$$

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Example

```
for (i = 0; i <= 99; ++i) {
    A[i];                    //reference a
    A[99-i];                 //reference b
    if (i >= 50)
        A[2*i];              //reference c
}
```

Assume: fully assiociative cache with line size 1

Calculate: $\text{FRD}_{a \to b}$

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Example

$$
\begin{aligned}
\text{map}_a &= \{i \to l \in I_a \times L \mid l = i\} \\
\text{map}_b &= \{i \to l \in I_b \times L \mid l = 99 - i\} \\
\text{map}_c &= \{i \to l \in I_c \times L \mid l = 2i\} \\
\text{iters}_{a\to a\to b} &= \{i, j \to k \in I_a \times I_b \times I_a \mid i < k \leq j\} \\
\text{iters}_{a\to b\to b} &= \{i, j \to k \in I_a \times I_b \times I_b \mid i \leq k < j\} \\
\text{iters}_{a\to c\to b} &= \{i, j \to k \in I_a \times I_b \times I_c \mid i \leq k < j\}
\end{aligned}
$$

$$
\begin{aligned}
\text{reuse}_{a\to b} = \{(i,j) \in I_a \times I_b \mid i \leq j \wedge i = 99 - j \wedge \\
\neg(\exists k \in I_c : i \leq k < j \wedge 2k - 100 = i)\}
\end{aligned}
$$

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Example

$$\text{FADS}_{a\to b} = \{(i,l) \mid (33 \le i < l \le -i + 99) \vee$$
$$(\exists \alpha : 2\alpha = 1 + i \wedge 1 \le i \le 31, -l + 99, l - 1) \vee$$
$$(\exists \alpha : 2\alpha = l \wedge 0 \le l \le -2i + 96 \wedge 33 \le i) \vee$$
$$(\exists \alpha, \beta : 2\alpha = 1 + i \wedge 2\beta = l \wedge 0 \le l < i \le 31)\}$$

$$\text{FRD}_{a\to b} =$$

$$(33 \le i \le 49 \mapsto 99 - 2i) +$$
$$(1 \le i \le 31 \mapsto [(-i - 1) \bmod 2 = 0] \cdot (99 - 2i)) +$$
$$(33 \le i \le 48 \mapsto 49 - i) +$$
$$\left(1 \le i \le 31 \mapsto [(-i - 1) \bmod 2 = 0] \cdot \left(\frac{p}{2} - \frac{p + 1 \bmod 2}{2} + \frac{1}{2}\right)\right)$$

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Complication

No-intervening-access constraint (1d):

$$\forall t \in \mathcal{R} : \neg \left(\exists \mathbf{k}_t \in I_t : \mathbf{i}_r \prec \mathbf{k}_t \prec \mathbf{j}_s \wedge t@\mathbf{k}_t = r@\mathbf{i}_r\right)$$

$\Rightarrow$ Omega sometimes fails to construct (accurate) reuse$_{r \to s}$
$\Rightarrow$ Equivalent to taking differences of sets
$\Rightarrow$ Doing this in PolyLib would be madness

Exacerbated by

- Many references
- More complicated references
  - Caches with limited associativity
  - Line sizes different from 1

  $\Rightarrow$ more existential variables

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Restricted Reuse Pairs

- $\text{reuse}^s_{r\to s}$: an iteration of reference r paired off with the first iteration of reference s that accesses the same memory element
  $\Rightarrow \text{reuse}^s_{r\to s}$ is a function
- $\text{reuse}^s_{r\to t\to s}$: an iteration of reference r paired off with the first iteration of reference s that accesses the same memory element, such that an intermediate iteration of reference t also accesses the same memory element
  ($\Rightarrow \text{reuse}^s_{r\to s\to s}$ is empty)

$$\text{reuse}_{r\to s} = \text{reuse}^s_{r\to s} \setminus \bigcup_t \text{reuse}^s_{r\to t\to s}$$

$$\text{reused}_{r\to s} = \text{reused}^s_{r\to s} \setminus \bigcup_t \text{reused}^s_{r\to t\to s}$$

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Restricted Reuse Pairs

$\forall r, s \in \mathcal{R}$: $\text{reuse}^s_{r\to s} = \left\{(\mathbf{i}_r, \mathbf{j}_s) \in \mathbb{Z}^{2n}\right.$: subject to conditions (2a)–(2d)$\left.\right\}$

$$\mathbf{i}_r \in I_r \wedge \mathbf{j}_s \in I_s \qquad \text{(\textit{iteration space})} \qquad (2a)$$

$$\mathbf{i}_r \prec \mathbf{i}_s \qquad \text{(\textit{execution ordering})} \qquad (2b)$$

$$r@\mathbf{i}_r = s@\mathbf{j}_s \qquad \text{(\textit{same location})} \qquad (2c)$$

$$\neg\left(\exists \mathbf{k}_s \in I_s : \mathbf{i}_r \prec \mathbf{k}_s \prec \mathbf{j}_s \wedge t@\mathbf{k}_s = r@\mathbf{i}_r\right)$$

(*no intervening access from* s) (2d)

$$\text{reuse}^s_{r\to t\to s} = \{(\mathbf{i}_r, \mathbf{j}_s) \in \text{reuse}^s_{r\to s} \mid$$
$$\exists \mathbf{k}_t \in I_t : \mathbf{i}_r \prec \mathbf{k}_t \prec \mathbf{j}_s \wedge t@\mathbf{k}_t = r@\mathbf{i}_r\}$$

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Forward Reuse Distance

$$\text{FRD}'_{r\to s} : \text{reused}^s_{r\to s} \to \mathbb{Z} : \mathbf{i}_r \mapsto \begin{cases} \left|\text{FADS}^s_{r\to s}\right| & \mathbf{i}_r \not\in \bigcup_t \text{reused}^s_{r\to t\to s} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{i}_r \in \text{reused}^s_{r\to s} \cap \text{reused}^t_{r\to t}$$
$$\wedge\ \text{FRD}'_{r\to s}(\mathbf{i}_r) \neq 0 \wedge \text{FRD}'_{r\to t}(\mathbf{i}_r) \neq 0 \Rightarrow s = t$$

$$\bigcup_{s\in\mathcal{R}} \text{reused}^s_{r\to s} = \bigcup_{s\in\mathcal{R}} \text{reused}_{r\to s}; \text{FADS}_{r\to s} = \text{FADS}^s_{r\to s} \cap \text{reused}_{r\to s} \times L$$

$$\Rightarrow \text{FRD}_r = \sum_{s\in\mathcal{R}} \text{FRD}'_{r\to s}$$

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Forward Reuse Distance

$$\text{FRD}'_{r\to s} : \text{reused}^s_{r\to s} \to \mathbb{Z} : \mathbf{i}_r \mapsto \begin{cases} \left|\text{FADS}^s_{r\to s}\right| & \mathbf{i}_r \notin \bigcup_t \text{reused}^s_{r\to t\to s} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{E}\left(\text{reused}^s_{r\to t\to s}; \mathbf{i}_r\right) : \text{reused}^s_{r\to s} \to \{0,1\} :$$

$$\mathbf{i}_r \mapsto \begin{cases} 1 & \mathbf{i}_r \in \bigcup_t \text{reused}^s_{r\to t\to s} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{FRD}'_{r\to s} = \mathcal{E}\left(\text{FADS}^s_{r\to s}; \mathbf{i}_r\right) \prod_{t\in\mathcal{R}} \left(1 - \mathcal{E}\left(\text{reused}^s_{r\to t\to s}; \mathbf{i}_r\right)\right)$$

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Example

$$\text{FADS}^b_{a\to b} = \{(i,l) \mid (0 \le i < l \le -i + 99) \vee$$
$$(\exists \alpha : 2\alpha = l \wedge 0 \le l \le -2i + 96, i)$$

$$\mathcal{E}(\text{FADS}^b_{a\to b}; i) = (0 \le i \le 49 \mapsto 99 - 2i) +$$
$$\begin{cases} 0 \le i \le 32 \mapsto \frac{p}{2} - \frac{p \bmod 2}{2} + 1 \\ 32 \le i \le 48 \mapsto 49 - i \end{cases}$$

$$\text{reused}^b_{a\to c\to b} = \{i \mid \exists \alpha : i = 2\alpha \wedge 0 \le i \le 32\}$$

$$\mathcal{E}(\text{reused}^b_{a\to c\to b}; i) = 0 \le i \le 32 \mapsto [(-i) \bmod 2 = 0] \cdot 1$$

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Calculating Restricted Reuse Pairs

- Omega/PolyLib
  Single difference $\Rightarrow$ problem alleviated
- PIP
  Consider

$$\text{reuse}^{\emptyset}_{r \to s} = \{(\mathbf{i}_r, \mathbf{j}_s) \in I_r \times I_s \mid \mathbf{i}_r \prec \mathbf{i}_s \wedge r@\mathbf{i}_r = s@\mathbf{j}_s\}$$

$$\text{reuse}^{s}_{r \to s} = \{(\mathbf{i}_r, \mathbf{j}_s) \mid \exists \mathbf{j}'_s : (\mathbf{i}_r, \mathbf{j}'_s) \in \text{reuse}^{\emptyset}_{r \to s} \wedge \mathbf{j}_s = \min_{\prec}(\text{reuse}^{\emptyset}_{r \to s}; \mathbf{i}_r)\}$$

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