Enumerating Integer Projections of Parametric Polytopes: A Direct Approach

Sven Verdoolaege, Maurice Bruynooghe

K.U.Leuven—Dept. of Computer Science

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Overview

- Enumerating Parametric Polytopes
  - Computation
  - Representation
- Integer Projections
  - Existential variable elimination
  - Slicing
  - Splitting
- To do

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Ehrhart Quasi-Polynomials

$$\begin{aligned} S &= \mathbb{Z}^d \cap \{\mathbf{x} \in \mathbb{Q}^d | A\mathbf{x} + C\mathbf{p} + \mathbf{b} \geq 0\} \\ &= \mathbb{Z}^d \cap \left\{\mathbf{x} \in \mathbb{Q}^d | V(\mathbf{p})\boldsymbol{\nu}, \boldsymbol{\nu} \geq \mathbf{0}, \sum \nu = 1\right\} \end{aligned}$$

Vertex $V_j(\mathbf{p}) = \sum_i \lambda_{ji} p_i + f_j$

$\#S$ is an Ehrhart quasi-polynomial $\mathcal{E}(\mathbf{p})$

Ehrhart Quasi-Polynomials

$$
\begin{aligned}
S &= \mathbb{Z}^d \cap \{\mathbf{x} \in \mathbb{Q}^d | A\mathbf{x} + C\mathbf{p} + \mathbf{b} \geq 0\} \\
&= \mathbb{Z}^d \cap \left\{\mathbf{x} \in \mathbb{Q}^d | V(\mathbf{p})\boldsymbol{\nu}, \boldsymbol{\nu} \geq \mathbf{0}, \sum \nu = 1\right\}
\end{aligned}
$$

Vertex $V_j(\mathbf{p}) = \sum_i \lambda_{ji} p_i + f_j$

$\#S$ is an Ehrhart quasi-polynomial $\mathcal{E}(\mathbf{p})$

- Quasi-polynomial of degree d in $\mathbf{p}$
- Coefficients: periodic numbers $u_{\mathbf{p}}$ with period $\mathbf{s}$

$$u_{\mathbf{p}} = \mathbf{u}[p_1 \bmod s_1][p_2 \bmod s_2] \ldots [p_n \bmod s_n]$$

- Period s_i is lcm of denominators of λ_{ji}
- Can be obtained through interpolation or directly

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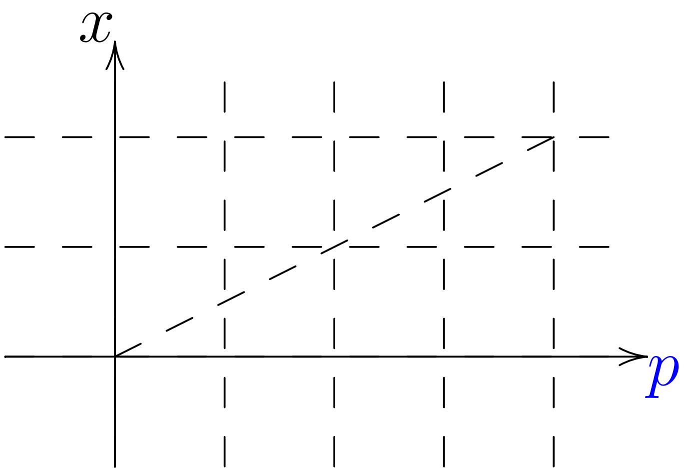
Interpolation Example

$$P = \{x | x \geq 0, 2x \leq p\} \qquad V = \left\{0, \frac{p}{2}\right\}$$



$$\begin{aligned}\mathcal{E}(p) &= [a, b]_p p + [c, d]_p \\ &= \begin{cases} ap + c & p \text{ even} \\ bp + d & p \text{ odd} \end{cases}\end{aligned}$$

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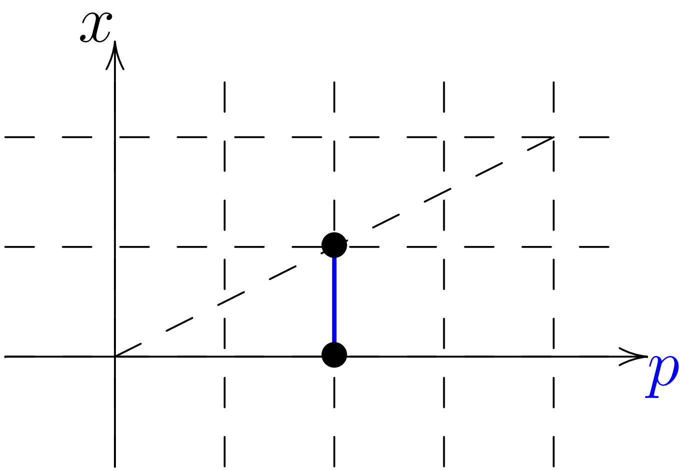

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$$2a + c = 2$$

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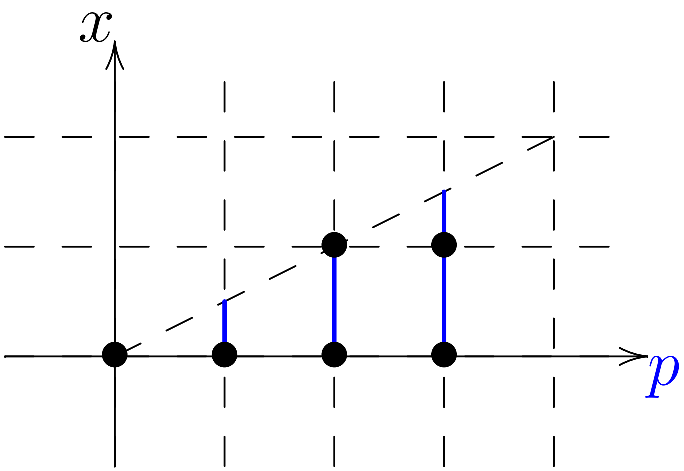

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$$\begin{aligned} c &= 1 \\ b + d &= 1 \\ 2a + c &= 2 \\ 3b + d &= 2 \end{aligned}$$

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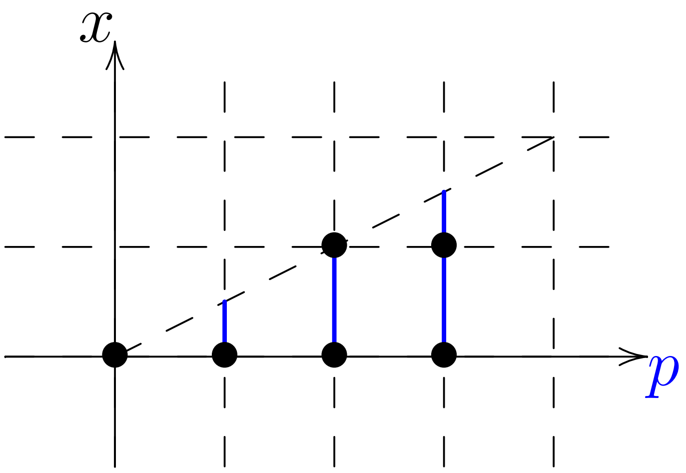

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$$\begin{aligned} c &= 1 \\ b + d &= 1 \\ 2a + c &= 2 \\ 3b + d &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= 1/2 \\ b &= 1/2 \\ c &= 1 \\ d &= 1/2 \end{aligned}$$

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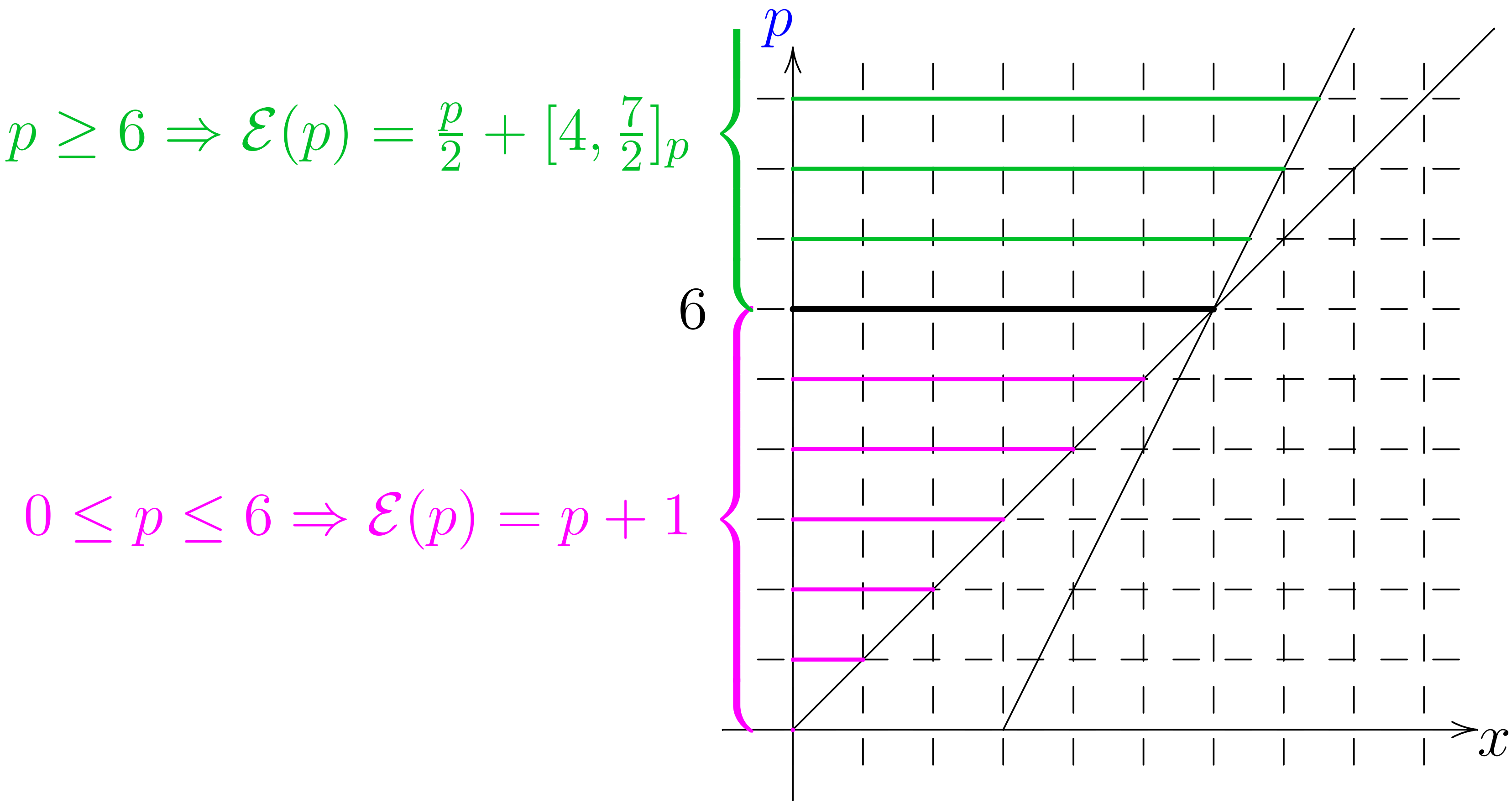
Validity domains

$$P = \{x | x \geq 0, 2x \leq p + 6, x \leq p\}$$



Enumerations

$$P = \{x | x \geq 0, 2x \leq p + 6, x \leq p\}$$

$$\mathcal{E}(P; p) = \begin{cases} 0 \leq p \leq 6 & \mapsto p + 1 \\ p \geq 6 & \mapsto \frac{p}{2} + [4, \frac{7}{2}]_p \end{cases}$$

Enumerations

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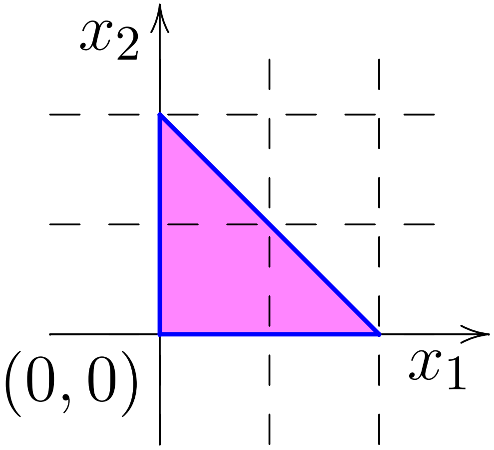
or

$$\mathcal{E}(P; p) = \begin{cases} 0 \leq p \leq 6 & \mapsto p + 1 \\ p \geq 6 & \mapsto \frac{p}{2} - \{\frac{p}{2}\} + 4 \end{cases}$$

Barvinok Example



$$T = \{\mathbf{x} | x_1 \geq 0 \wedge x_2 \geq 0 \wedge x_1 + x_2 \leq 2\}$$

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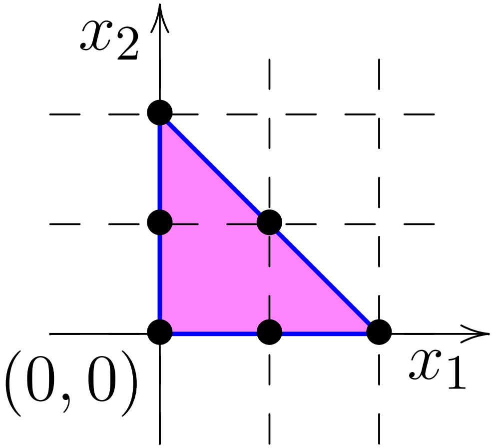
Barvinok Example



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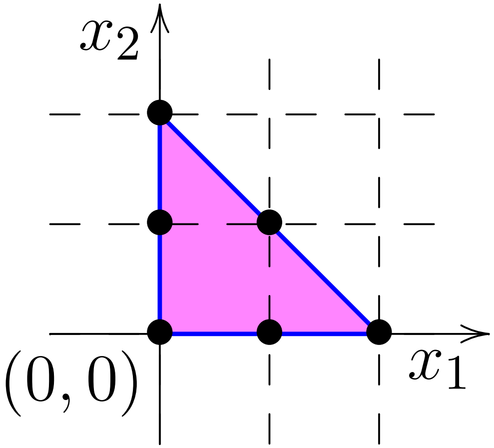

$$T = \{\mathbf{x} | x_1 \geq 0 \wedge x_2 \geq 0 \wedge x_1 + x_2 \leq 2\}$$

Generating function: $f(P;\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in P \cap \mathbb{Z}^d} \mathbf{x}^{\boldsymbol{\alpha}}$

$$f(T;\mathbf{x}) = 1 + x_1 + x_1^2 + x_2 + x_1 x_2 + x_2^2$$

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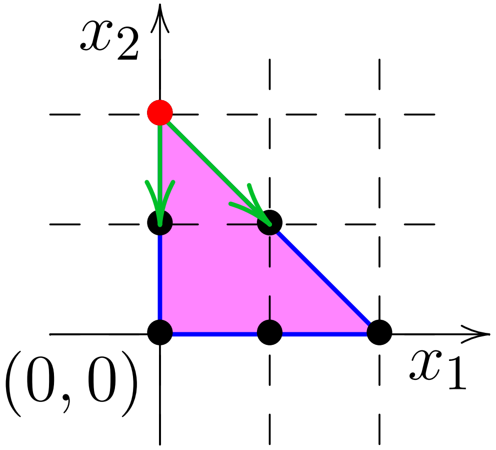
Barvinok Example



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$f(T; \mathbf{x}) = 1 + x_1 + x_1^2 + x_2 + x_1 x_2 + x_2^2$ or

$$\frac{x_2^2}{(1 - x_2^{-1})(1 - x_1 x_2^{-1})}$$

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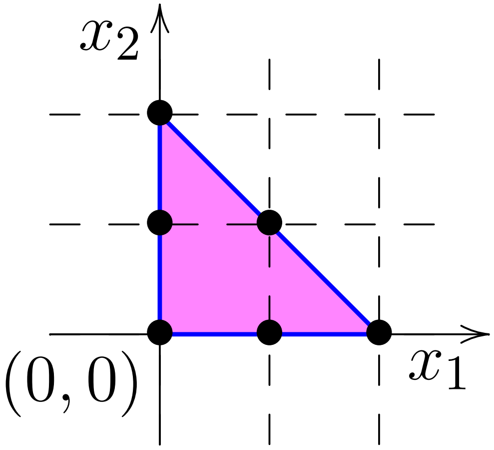
Barvinok Example



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$f(T;\mathbf{x}) = 1 + x_1 + x_1^2 + x_2 + x_1x_2 + x_2^2$ or

$$\frac{x_2^2}{(1-x_2^{-1})(1-x_1x_2^{-1})} + \frac{x_1^2}{(1-x_1^{-1})(1-x_1^{-1}x_2)} + \frac{1}{(1-x_1)(1-x_2)}$$

Number of lattice points: $f(T;\mathbf{1}) = 6$

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Barvinok's Algorithm

1. For each vertex $\mathbf{v}_i$ of P
    - (a) Determine supporting cone $\operatorname{cone}(P, \mathbf{v}_i)$
    - (b) Let $K = \operatorname{cone}(P, \mathbf{v}_i) - \mathbf{v}_i$
    - (c) Decompose K into unimodular cones $\{\, \epsilon_j, K_j \,\}$
    - (d) For each K_j
        - i. Determine $f(K_j; \mathbf{x})$
    - (e) $f(\operatorname{cone}(P, \mathbf{v}_i); \mathbf{x}) = \sum_j \epsilon_j \mathbf{x}^{E(\mathbf{v}_i, K_j)} f(K_j; \mathbf{x})$
2. $f(P; \mathbf{x}) = \sum_i f(\operatorname{cone}(P, \mathbf{v}_i); \mathbf{x})$
3. evaluate $f(P; \mathbf{1})$

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Parametric Barvinok

1. For each vertex $\mathbf{v}_i(\mathbf{p})$ of P
   - (a) Determine supporting cone $\operatorname{cone}(P, \mathbf{v}_i(\mathbf{p}))$
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   - (e) $f(\operatorname{cone}(P, \mathbf{v}_i(\mathbf{p})); \mathbf{x}) = \sum_j \epsilon_j \mathbf{x}^{E(\mathbf{v}_i(\mathbf{p}), K_j)} f(K_j; \mathbf{x})$
2. For each validity domain D of P
   - (a) $f(P; \mathbf{x}) = \sum_{\mathbf{v}_i \in D} f(\operatorname{cone}(P, \mathbf{v}_i(\mathbf{p})); \mathbf{x})$
   - (b) evaluate $f(P; \mathbf{1})$

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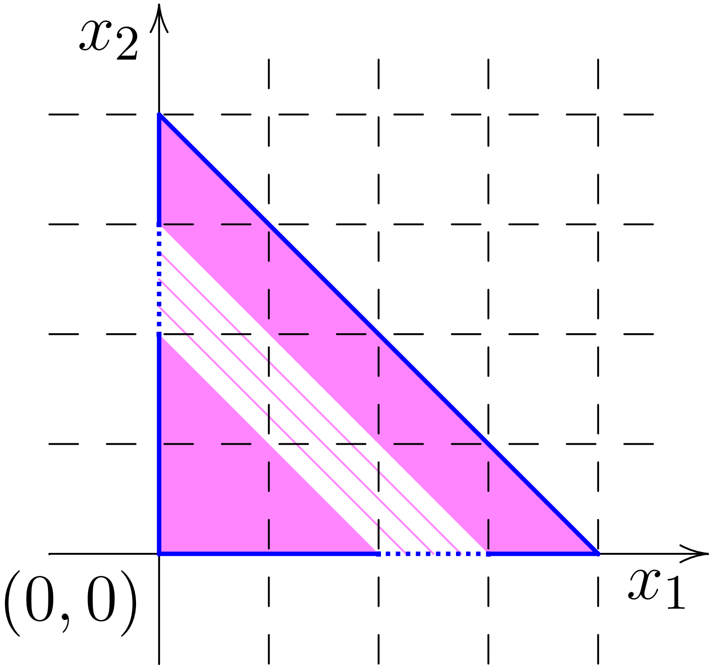
The Lattice Point Near a Vertex

$$E(\mathbf{v}_i(\mathbf{p}), K_j) = \sum_k \lceil \lambda_k(\mathbf{p}) \rceil B_{jk}$$

$$\lceil \lambda_k(\mathbf{p}) \rceil = \lambda_k(\mathbf{p}) + \{-\lambda_k(\mathbf{p})\}$$

⇒ Expressions with fractional parts instead of lookup-tables

Parametric Barvinok Example



$$\{\mathbf{x} | x_1 \geq 0 \wedge x_2 \geq 0 \wedge 2x_1 + 2x_2 \leq p\}$$

$$\frac{x_2^{\frac{p}{2}-\{\frac{p}{2}\}}}{(1-x_2^{-1})(1-x_1x_2^{-1})} + \frac{x_1^{\frac{p}{2}-\{\frac{p}{2}\}}}{(1-x_1^{-1})(1-x_1^{-1}x_2)} + \frac{1}{(1-x_1)(1-x_2)}$$

$$f(T;\mathbf{1}) = \frac{1}{8}p^2 + \left(\frac{3}{4} - \frac{1}{2}\{\frac{p}{2}\}\right) p + \frac{1}{2}\left(\{\frac{p}{2}\}^2 - \frac{3}{2}\{\frac{p}{2}\}\right) + 1$$

37.0%

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Lookup-tables vs. Fractional Parts

$$S = \{\, t \mid 0 \le i \wedge 1024t - 39800 \le i \le 199 \wedge 0 \le k \le 198 \\ \wedge 0 \le j \le 199 \wedge i + 200k \le 823 + 1024t \,\}$$

Vertices:

$$\frac{i}{1024} + 25\frac{k}{128} - \frac{823}{1024} \qquad \text{and} \qquad \frac{i}{1024} + \frac{4975}{128}$$

$\Rightarrow$ Lookup-tables of size $1024 \cdot 128$ (exponential in input size)

Lookup-tables vs. Fractional Parts

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$\Rightarrow$ Lookup-tables of size $1024 \cdot 128$ (exponential in input size)

Using fractional parts:

$$-\frac{25}{128}k - \left\{\frac{i + 888}{1024}\right\} + \left\{\frac{i + 200k + 200}{1024}\right\} + \frac{2539}{64}$$

Ingredients of Enumerations

- Fractional part

$$\{x\} = x - \lfloor x \rfloor$$

E.g., $\left\{-\frac{27}{20}\right\} = \frac{13}{20}$

- Fractional map

$$f : \mathbb{Z}^n \to \mathbb{Q} : (x_1, \ldots, x_n) \mapsto \{a_0 + a_1 x_1 + \cdots + a_n x_n\}$$

with $a_0, a_1, \ldots, a_n \in \mathbb{Q}$
E.g.,

$$\left\{\frac{1}{8} i + \frac{7}{8}\right\}$$

$\mathbb{Q}\{\mathbf{p}\}$ is the set of all such maps over $\mathbf{p}$

Ingredients of Enumerations

- Periodic numbers

  Polynomials over the rationals in the fractional maps

$$\mathbb{P}\left\{\mathbf{p}\right\} = \mathbb{Q}[\mathbb{Q}\left\{\mathbf{p}\right\}]$$

E.g.,

$$-\left\{\frac{1}{8}i + \frac{1}{2}\right\} + \left\{\frac{1}{8}i + \frac{7}{8}\right\} - \frac{3}{8}$$

Ingredients of Enumerations

- Ehrhart quasi-polynomials

  Polynomials over the periodic numbers

$$\mathbb{E}[\mathbf{p}] = (\mathbb{P}\{\mathbf{p}\})[\mathbf{p}]$$

E.g.,

$$\frac{1}{4}i - \left\{\frac{1}{8}i + \frac{1}{2}\right\} + \left\{\frac{1}{8}i + \frac{7}{8}\right\} - \frac{3}{8}$$

Ingredients of Enumerations

- Strides

$$E' = [f = 0] \cdot E_1 + [f \neq 0] \cdot E_2$$

with $f \in \mathbb{Q}\{\mathbf{p}\}$

$$E'(\mathbf{p}) = \begin{cases} E_1(\mathbf{p}) & \text{if } f(\mathbf{p}) = 0 \\ E_2(\mathbf{p}) & \text{otherwise} \end{cases}$$

Equivalent to

$$E_1 + \left(\{F\} - \left\{ F - \frac{1}{m} \right\} + \frac{m-1}{m} \right) (E_2 - E_1)$$

with $f = \{F\}$ and m the common denominator of F

Ingredients of Enumerations

- Strides

  E.g.,

$$\left[\left\{\frac{1}{4}i\right\} = 0\right] \cdot \left(\frac{1}{4}i - \left\{\frac{1}{8}i + \frac{1}{2}\right\} + \left\{\frac{1}{8}i + \frac{7}{8}\right\} - \frac{3}{8}\right)$$

  Equal to:

$$\left[\frac{1}{4}, 0, 0, 0\right]_i i + 0$$

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Ingredients of Enumerations

- Enumeration

  Pairs of validity domains and Ehrhart quasi-polynomials

$$\mathcal{E}_{\mathbf{p}} = \left\{ (D, E) \in 2^{\mathbb{Z}^n} \times \mathbb{E}[\mathbf{p}] \right\}$$

$$\mathcal{E}(\mathbf{p}) = \begin{cases} E_i(\mathbf{p}) & \text{if } \exists i : \mathbf{p} \in D_i \\ 0 & \text{otherwise} \end{cases}$$

  Initially, the D are convex polytopes, but they can also be unions of convex polytopes or other generalizations.

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62.9%

Integer Projections

Extended problem:
Find the number of points in the integer projection of a rational parametric polytope:

$$\#\pi_d \left(\mathbb{Z}^{(d+d')} \cap \left\{ (\mathbf{y}, \boldsymbol{\epsilon}) \in \mathbb{Q}^{(d+d')} \mid A\mathbf{y} + D\boldsymbol{\epsilon} + C\mathbf{p} + \mathbf{b} \geq 0 \right\} \right)$$

or

$$\# \left\{ \mathbf{y} \in \mathbb{Z}^d \mid \exists \boldsymbol{\epsilon} \in \mathbb{Z}^{d'} : A\mathbf{y} + D\boldsymbol{\epsilon} + C\mathbf{p} + \mathbf{b} \geq 0 \right\}$$

Notation:

$$\mathbf{x} = (\mathbf{y}, \boldsymbol{\epsilon}, \mathbf{p})$$

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Negative Constraint

Take a lower and an upper bound

$$n_l \epsilon_i + \langle \mathbf{c}_l, \mathbf{x} \rangle + c_l \geq 0$$

$$-n_u \epsilon_i + \langle \mathbf{c}_u, \mathbf{x} \rangle + c_u \geq 0$$

$\Rightarrow$

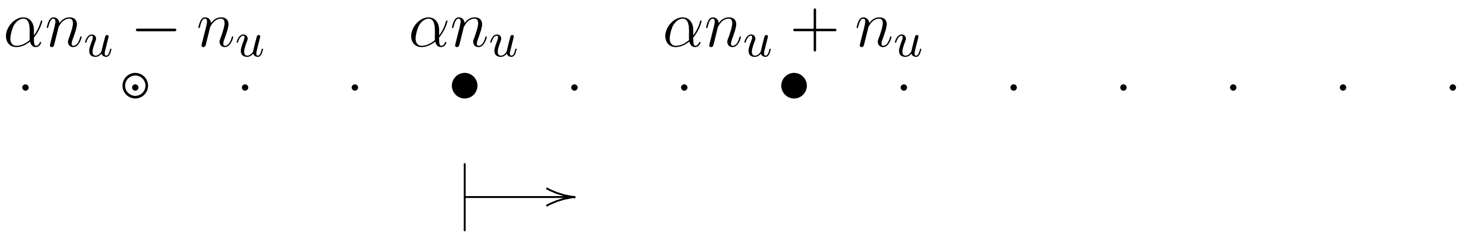
$$-n_u(\langle \mathbf{c}_l, \mathbf{x} \rangle + c_l) \leq n_u n_l \epsilon_i \leq n_l(\langle \mathbf{c}_u, \mathbf{x} \rangle + c_u)$$

$$P \models n_l(\langle \mathbf{c}_u, \mathbf{x} \rangle + c_u) + n_u(\langle \mathbf{c}_l, \mathbf{x} \rangle + c_l) + 1 \leq n_u n_l$$

$\Rightarrow$ at most one (integer) value for ϵ_i for each value of $\mathbf{x}$ (“negative” constraint)

Positive Constraint

$$n_u(n_l\epsilon_i) \geq n_u\alpha \Leftrightarrow n_u(n_l\epsilon_i) \geq n_u\alpha - n_u + 1$$



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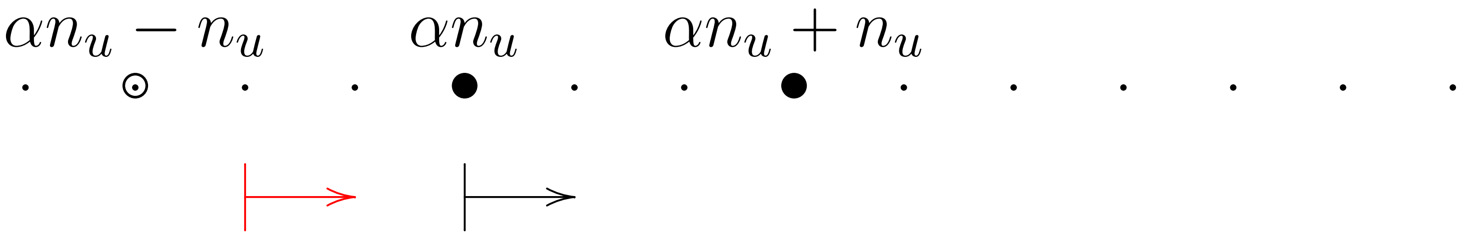

$\alpha n_u - n_u$

αn_u

$\alpha n_u + n_u$

Positive Constraint

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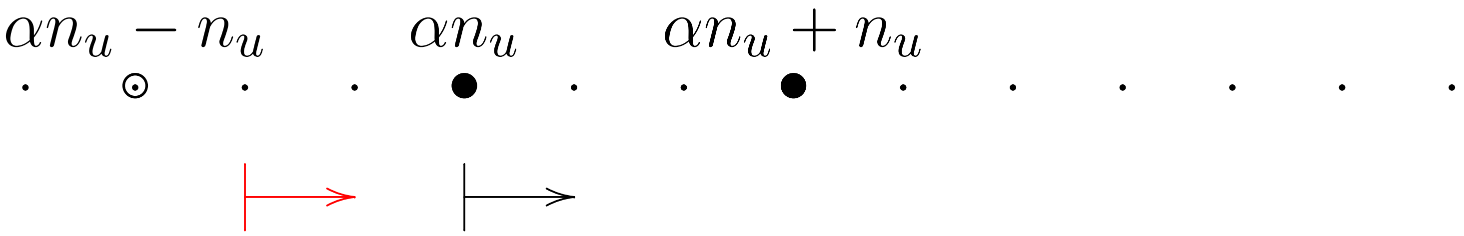

$$-n_u(\langle \mathbf{c}_l, \mathbf{x}\rangle + c_l) \leq n_u n_l \epsilon_i \leq n_l(\langle \mathbf{c}_u, \mathbf{x}\rangle + c_u)$$

$$\Updownarrow$$

$$-n_u(\langle \mathbf{c}_l, \mathbf{x}\rangle + c_l) - n_u + 1 \leq n_u n_l \epsilon_i \leq n_l(\langle \mathbf{c}_u, \mathbf{x}\rangle + c_u) + n_l - 1$$

$$P \models n_l(\langle \mathbf{c}_u, \mathbf{x}\rangle + c_u) + n_u(\langle \mathbf{c}_l, \mathbf{x}\rangle + c_l) + n_l - 1 + n_u - 1 + 1 \geq n_u n_l$$

$\Rightarrow$ at least one (integer) value for ϵ_i for each value of $\mathbf{x}$ (“positive” constraint $\equiv$ Omega test)

Eliminating Existential Variables

- Existential variable ϵ_i only yields positive constraints
  $\Rightarrow$ Eliminate ϵ_i

$$\#P(\mathbf{y};\boldsymbol{\epsilon};\mathbf{p}) = \#P(\mathbf{y};\epsilon_1,\ldots,\epsilon_{i-1},\epsilon_{i+1},\ldots,\epsilon_{d'};\mathbf{p})$$

75.3%

Eliminating Existential Variables

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  $\Rightarrow$ Eliminate ϵ_i

$$\#P(\mathbf{y};\boldsymbol{\epsilon};\mathbf{p}) = \#P(\mathbf{y};\epsilon_1,\ldots,\epsilon_{i-1},\epsilon_{i+1},\ldots,\epsilon_{d'};\mathbf{p})$$

- Existential variable ϵ_i yields a negative constraint and one bound is independent of the other ϵ_j
  $\Rightarrow$ ϵ_i will never be counted twice
  $\Rightarrow$ Remove existential quantification

$$\#P(\mathbf{y};\boldsymbol{\epsilon};\mathbf{p}) = \#P(\mathbf{y},\epsilon_i;\epsilon_1,\ldots,\epsilon_{i-1},\epsilon_{i+1},\ldots,\epsilon_{d'};\mathbf{p})$$

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If P contains a single existential variable, then it can always be handled this way.

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Line Removal

$$\overline{P} = \left\{ (\mathbf{y}, \boldsymbol{\epsilon}, \mathbf{p}) \in \mathbb{Q}^{(d+d'+n)} \mid A\mathbf{y} + D\boldsymbol{\epsilon} + C\mathbf{p} + \mathbf{b} \geq 0 \right\}$$

Suppose $\overline{P}$ contains a line $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, i.e.,

$$(\mathbf{y}, \boldsymbol{\epsilon}, \mathbf{p}) \in \overline{P} \Leftrightarrow (\mathbf{y} + \mathbf{a}, \boldsymbol{\epsilon} + \mathbf{b}, \mathbf{p} + \mathbf{c}) \in \overline{P}$$

or

$$\mathcal{E}(P; \mathbf{p}) = \mathcal{E}(P; \mathbf{p} + \mathbf{c})$$

Suppose $\mathbf{c} = \mathbf{0}$

- $\mathbf{a} \neq \mathbf{0} \Rightarrow P$ is unbounded polyhedron
- $\mathbf{b} \neq \mathbf{0} \Rightarrow$ eliminated

$\Rightarrow \mathbf{c} \neq \mathbf{0}$

81.4%

Line Removal

Assume $c_1 > 0$
Let

$$P'' = P|_{0 \le p_1 \le c_1 - 1}$$

$$\mathcal{E}(P; \mathbf{p}) = \mathcal{E}(P''; \mathbf{p} - \left\lfloor \frac{p_1}{c_1} \right\rfloor \mathbf{c})$$

$$\mathcal{E}_{P'';\mathbf{p}} = \{\, (D''_i, E''_i) \,\} \qquad \mathcal{E}_{P;\mathbf{p}} = \{\, (D_i, E_i) \,\}$$

$$D_i = \bigcup_{j=-\infty}^{\infty} D''_i + j\mathbf{c} \qquad E_i(\mathbf{p}) = E''_i(\mathbf{p} - j\mathbf{c})$$

⇒ Extra variables $q, \mathbf{r}$ in domains: $\mathbf{p} = q\mathbf{c} + \mathbf{r}, 0 \le r_1 \le c_1 - 1$

Overview

- Enumerating Parametric Polytopes
  - Computation
  - Representation
- Integer Projections
  - Existential variable elimination
  - Slicing
  - Splitting
- To do

85.1%

Splitting

If none of the previous cases apply, we can try to split the polytope.

$$\overline{P} = \overline{P}_- \cup \overline{P}_+ = \big(\overline{P} \cap \{\, \mathbf{x} \mid \langle \mathbf{c}, \mathbf{x} \rangle \le 0 \,\}\big) \cup \big(\overline{P} \cap \{\, \mathbf{x} \mid \langle \mathbf{c}, \mathbf{x} \rangle \ge 1 \,\}\big)$$

- $\mathbf{c} = (\mathbf{a}, \mathbf{0}, \mathbf{c}')$
  $\Rightarrow \mathcal{E}(P) = \mathcal{E}(P_-) + \mathcal{E}(P_+)$
- $d = 0 \Rightarrow \mathcal{E}(P; \mathbf{p}) = \left[\exists \boldsymbol{\epsilon} \in \mathbb{Z}^{d'} : (\boldsymbol{\epsilon}, \mathbf{p}) \in \overline{P}\right]$
  $\Rightarrow \mathcal{E}(P) = \mathcal{E}(P_-) + \mathcal{E}(P_+) - \mathcal{E}(P_-)\mathcal{E}(P_+)$
- Otherwise
  $\Rightarrow$ Calculate $\mathcal{E}(P; \mathbf{y}, \mathbf{p})$
  $\Rightarrow \mathcal{E}(P; \mathbf{p}) = \sum_{\mathbf{y}} \mathcal{E}(P; \mathbf{y}, \mathbf{p})$

88.8%

Summing an Enumeration

- Rewrite each Ehrhart quasi-polynomial as a polynomial in floor functions using $\{L\} = L - \lfloor L \rfloor$

$$E = \sum_{i,j,k} c_{i,j,k} \mathbf{y}^{\boldsymbol{\alpha}_i} \mathbf{p}^{\boldsymbol{\beta}_j} \mathbf{r}^{\gamma_k}$$

with $r_l = \lfloor a_0 + \sum_i a_i p_i + \sum_j b_j y_j \rfloor$ and $P \models r_l \geq 0$

- Consider each term $c\mathbf{y}^{\boldsymbol{\alpha}}\mathbf{p}^{\boldsymbol{\beta}}\mathbf{r}^{\gamma}$
  - Introduce new variables v_i for each factor in $\mathbf{y}^{\boldsymbol{\alpha}}$
  - Add constraints $1 \leq v_i \leq y_{m_i}$ to $D \Rightarrow D'$
  - Introduce new variables w_j for each factor in $\mathbf{r}^{\gamma}$
  - Add constraints $1 \leq w_j \leq r_{m_j}$ to $D' \Rightarrow D''$
  - Compute $c\mathbf{p}^{\boldsymbol{\beta}} \mathcal{E}(D''; \mathbf{p})$

92.5%

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Remaining Issues

- Split on which hyperplane ?
  - Constraint that is neither negative or positive ?
  - Coordinate of a parametric vertex ?
  - . . .
- Complexity ?
- Simplification of fractional part expressions

Related Work

- Clauss (1997)
  ⇒ Number of points in hull minus number of holes
  ⇒ Number of holes is equal to number of roots of some Ehrhart quasi-polynomial
- Seghir (2002)
  ⇒ Number of points in “fat facet”
  ⇒ Only works for a single existential variable
- Barvinok and Woods (2002)
  ⇒ Works directly on generating functions
  ⇒ Not implemented yet ?
  ⇒ Can it be extended to parametric case ?
- Pugh (1994)
  ⇒ Underspecified and apparently not implemented
  ⇒ Resorts to massive “splintering”